Mr Tomkins and the World Where Four Matters

A story for curious minds — in the tradition of Gamow's Mr Tompkins

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Prologue: The Lecture

Mr Tomkins had fallen asleep again.

He was sitting in the back row of a university physics lecture — he had wandered in from the rain, purely to get warm — and the professor at the front was writing something on the board that looked like a table made of curly brackets. Mr Tomkins' eyelids grew heavy. The words "recoupling of four representations" drifted past him like smoke.

He slept.

Chapter One: The Island of Threes

Mr Tomkins woke up on a beach.

The sand was white and fine, and the sky was a deep, comfortable blue. A girl of about fourteen was sitting next to him, building something in the sand. She had a bucket and spade and the focused expression of someone who is doing mathematics while pretending to make a sandcastle.

"Hello," said Mr Tomkins. "Where am I?"

"The Island of Threes," said the girl, without looking up. Her name, it turned out, was Polly. "Everything here works by combining things in groups of three. It's very tidy."

Mr Tomkins looked around. The beach was dotted with small tables, each with exactly three objects on it — three shells, three stones, three coloured tiles.

"What's tidy about three?" he asked.

Polly put down her spade. "When you combine things in groups of three, there's never any ambiguity about the order. Watch."

She picked up three shells — a red one, a blue one, and a green one.

"If I want to combine them all into one thing, I can do it two ways. Red-and-blue first, then add green." She held the red and blue shells together in her left hand, then pressed the green one against them with her right. "Or blue-and-green first, then add red." She separated them and tried again.

"And?" said Mr Tomkins.

"And I get exactly the same thing, both times." She opened her hands. In each case, a single smooth white pebble had appeared — identical, indistinguishable. "The order doesn't matter. Every possible bracketing gives the same result. That's called associativity."

Mr Tomkins nodded slowly. "So with three objects, you can't go wrong."

"You can never go wrong. There's only one answer, regardless of the order you combine them. The professors call the rule for doing it the Clebsch-Gordan rule — it tells you how to merge two things into one. But with three objects you just apply it twice, and the order you apply it doesn't matter. The Island of Threes is very peaceful."

She gestured at the beach. Groups of children were combining things — three kites, three musical notes, three colours of light — and every group looked serene, because every group always got the same answer.

"It sounds perfect," said Mr Tomkins.

"It is," said Polly. "But it's also a bit boring."

Chapter Two: Across the Water

Polly pointed to a dark smudge on the horizon.

"That's the Island of Fours. Things are different there."

"Different how?"

"Come and I'll show you." She produced a small boat from behind a rock (in dreams, this is perfectly normal), and they rowed across.

As they approached, Mr Tomkins noticed that the sky over the second island was slightly more interesting — a deeper blue, with a faint shimmer, as if more were happening there.

They beached the boat and walked up the shore. Here the tables had four objects each — four shells, four stones, four tiles.

A boy about Polly's age was sitting at the nearest table, looking very frustrated. His name, he said, was Raj.

"I'm trying to combine these four shells," said Raj, "and I keep getting different answers depending on the order I do it in."

He had four shells in front of him: red, blue, green, and yellow.

"Watch," he said. "I'll try Scheme One. Red and blue first, then green and yellow, then combine the pairs."

He picked up red and blue, and they merged into a single shell with a pale purple shimmer. Then he picked up green and yellow, and they merged into a shell with a golden-green shimmer. Then he pressed the two merged shells together.

A new shell appeared: deep violet with gold flecks.

"Now Scheme Two. Red and yellow first, then blue and green, then combine."

He tried it. Red and yellow made a warm orange shell. Blue and green made a cool teal shell. Pressing those together gave — a shell that was clearly related to the first result but not identical. It was violet-gold too, but the violet was deeper and the gold was brighter.

"They're not the same!" said Mr Tomkins.

"Exactly," said Raj. "And there's a third way too — red and green first, then blue and yellow. That gives yet another violet-gold shell. They're all in the same family, but they're not identical. The order matters."

Polly had sat down next to Raj. "This is the new thing that four introduces. With three objects, all orders give the same result. With four objects, you get different results depending on the order, and you need a special rule — a special number, really — to convert between them."

"What's the number?" asked Mr Tomkins.

Raj picked up a stick and drew in the sand. He drew a tetrahedron — a triangular pyramid with four vertices, four triangular faces, and six edges connecting the vertices.

"The four shells sit at the four vertices," he said. "The way you combine any two of them is written on the edge between them. There are six pairs you can make from four shells" — he counted the edges — "and the amount by which Scheme One and Scheme Three disagree is written in all six edges at once. The whole tetrahedron carries the conversion rule."

"And the conversion rule has a name?" asked Mr Tomkins.

"The 6j symbol," said Polly. "Six because it has six entries — one for each edge of the tetrahedron. j because each entry is a spin — a kind of rotation number. It's one number built from six ingredients, all arranged around a tetrahedron."

"And this 6j symbol tells you how different Scheme One is from Scheme Three?"

"Exactly that," said Raj. "And it's always a number between minus one and plus one. For some types of objects it's minus one-third. That same minus-one-third number appears when physicists recalculate nuclear forces, when they study exotic particles made of quarks, and even when they study how quantum computers process information. The same tetrahedron, the same number, everywhere."

Mr Tomkins looked at the tetrahedron in the sand. "So the universe needs this extra number — this 6j symbol — only because four objects are being combined?"

"Only because four," said Polly. "With three, you never need it. With four, you always do — if the objects have any interesting internal structure, any spin or charge or flavour. The moment you ask what happens when four things interact, a tetrahedron appears whether you invited it or not."

Chapter Three: The Frustrated Financial Wizard

Further up the beach they found an elderly man in a pinstriped suit, sitting at a large table on which were arranged four small cardboard boxes labelled **A**, **B**, **C**, and **D**. He was staring at the boxes with an expression of profound vexation.

"I'm ruined," he said, to no one in particular.

"What happened?" asked Mr Tomkins.

"I'm a financial risk manager," said the man, who gave his name as Mr Vier. "I run a bank. Or rather, I used to run a bank. I had four clients — call them A, B, C, and D. Each pair of clients did business together. I measured how much risk each pair of them shared." He pointed to the edges of an imaginary tetrahedron in the air. "Six pairs, six risk numbers."

"That sounds very sensible," said Mr Tomkins.

"It sounds sensible. But here is what happened. I measured the risk between A and B. Fine. I measured the risk between B and C. Fine. I measured the risk between A and C. Fine. Each pair, separately, looked consistent. But when all four clients had a problem at the same time —" He shuddered. "The whole system collapsed in a way that none of my pairwise measurements predicted."

Polly leaned forward. "What went wrong is that you only checked the edges. But the tetrahedron has faces too."

"Faces?" said Mr Vier.

"Four triangular faces," said Raj, drawing in the sand again. "Each face has three edges. Each face has its own consistency condition — you call it a pricing residual, or an arbitrage condition. If three of your clients are all connected, the three pairwise risks between them have to satisfy a triangle rule: they can't be mutually inconsistent."

"I did check triangles," said Mr Vier. "Every trio of clients was triangle-consistent."

"But there are four trios," said Polly. "And the four triangular residuals have their own global consistency condition — the condition that they fit together around the hollow tetrahedron without gaps or overlaps. That's what you didn't check."

She wrote an equation in the sand:

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residual(BCD) − residual(ACD) + residual(ABD) − residual(ABC) = 0
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"If this equation fails," she said, "then your four clients have risks that are individually pairwise consistent, triangle consistent, but globally incoherent. No bilateral contract can fix it, because bilateral contracts live on edges, and the problem lives on the faces."

Mr Vier stared at the equation. "I had four clients and I missed a global consistency condition that only appears when all four are considered together."

"Exactly," said Polly. "Three clients can always be made consistent. Four clients, for the first time, might not be — and the thing that detects the inconsistency lives on the faces of the tetrahedron they form. You need all four simultaneously."

"Nobody taught me this," said Mr Vier, quietly.

"Nobody teaches it," said Raj. "But it was known. Mathematicians call it the second cohomology group. When it's non-zero, there's a global obstruction that you can't see from any smaller vantage point. No amount of pairwise checking can find it."

Mr Vier picked up his briefcase, which was entirely empty, and walked away down the beach looking thoughtful. Mr Tomkins watched him go.

"That happens a lot," said Polly. "Banks discover this rule the hard way."

Chapter Four: The Pentagon Game

They walked inland and came to a clearing where a group of children were playing a game. Five of them stood at the five corners of a large pentagon drawn in the chalk on the ground. Each player held a coloured card.

The rules, as a referee explained to Mr Tomkins, were as follows.

At each corner of the pentagon, a player had to choose one of two options — call them 0 and 1. At each edge of the pentagon, there was a rule relating the choices of the two players at its endpoints. The whole pentagon's rules had to be consistent: you had to be able to find a choice for each player such that all five edge-rules were satisfied simultaneously.

"Can they always win?" asked Mr Tomkins.

"That depends," said the referee, "on whether the five 6j symbols around the pentagon satisfy the Pentagon Equation."

She drew the relevant diagram — a pentagon with a smaller pentagon inside it, connected by lines to form a star.

"Imagine that each edge of this pentagon is a 6j symbol — a conversion rule between two ways of combining four objects. Going around the pentagon clockwise, you apply five such conversions in sequence. If the 6j symbols are consistent — if the pentagon equation holds — then after five steps you're back where you started: the result of going all the way around is the identity. Everything checks out, and the players can always win."

"And if the pentagon equation fails?" asked Mr Tomkins.

"Then the game is impossible. You can satisfy four of the five edge-rules, but never all five simultaneously. The contradiction lives in the global loop, not in any individual edge."

Mr Tomkins watched the children play. One team moved confidently around the pentagon, placing their cards in a pattern that satisfied every rule. The referee nodded approval.

"These children are playing the game for quantum states," said the referee. "Their 6j symbols come from a fusion category — a mathematical structure describing how particles can be combined and split. Because their 6j symbols satisfy the pentagon equation, there's no contradiction, and they can win."

"What kinds of physics use the pentagon equation?" asked Raj.

"Everything that involves combining and splitting in sequence," said the referee. "Topological phases of matter — special materials where particles called anyons can braid around each other. Conformal field theories — the mathematics of the early universe and of string theory. Knot theory — the mathematics of how ropes can be tangled. Quantum error correction — how to store quantum information safely. In every case, the pentagon equation is the master consistency condition. If it fails, the theory is inconsistent. If it holds, the theory is self-coherent."

"And all of these are about combining four things?" asked Mr Tomkins.

"The pentagon equation is about five 6j symbols," said the referee, "and each 6j symbol is about four objects. So yes: the pentagon equation is the self-consistency condition for a world in which four-object coupling is fundamental. It says: no matter which sequence of reorderings you perform, as long as you end up in the same final state, you get the same answer. The universe doesn't care about the path, only the destination."

Chapter Five: The Particle Physicist's Canteen

Mr Tomkins and Polly followed the smell of coffee to a beach canteen, where a physicist was eating lunch and looking very pleased with herself.

"I solved the nuclear force problem," she announced, before they had ordered anything.

Her name was Dr Pandya — or rather, she was explaining the work of a Dr Pandya from many decades past, who had noticed something remarkable. When you had a nucleus with a proton and a hole (a missing neutron) in certain shells, you could describe it in two ways.

"You could say: particle A plus particle B, combined with intermediate angular momentum J, then that combined thing interacts with particle C." She stirred her coffee. "Or you could say: particle A minus the hole of B — which is the same as A plus the antiparticle of B — combined with a different intermediate J'."

"And the two descriptions give the same physics?" asked Mr Tomkins.

"They describe the same physical nucleus, yes. But converting between the two descriptions requires the 6j symbol." She drew the recoupling formula on a napkin:

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Ṽ(ja, jc; J) = −Σ (2J'+1) × {ja jb J / jc jd J'} × V(ja, jb; J')
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"The curly bracket is the 6j symbol. The sum over J' converts one description into the other. And the number that comes out for the simplest interesting case — all four spins equal to 1 — is minus one-third."

"The same minus one-third that Raj mentioned?" asked Mr Tomkins.

"The very same." She smiled. "That minus one-third also appears when you calculate the properties of the X(3872) — an exotic particle made of a charm quark and an anti-charm quark and two other quarks, sitting just on the boundary of whether it should exist at all. The same tetrahedron. The same amplitude. Completely different physics, completely identical mathematics."

"Why?" asked Mr Tomkins.

"Because in both cases, you are asking: what is the amplitude for rearranging the coupling order of four things with spin one? And the universe only has one answer to that question. The answer lives on the tetrahedron, and the tetrahedron doesn't know whether you're talking about nucleons or quarks."

Chapter Six: The Quantum Computer's Trouble with Triangles

Outside the canteen, a teenager was struggling with a small box covered in wires and lights.

"My quantum computer keeps making errors," he said. His name was Kieran. "I'm trying to store information in quantum bits — qubits. I have a block of three qubits, and I need to combine them in a specific way to make a logical qubit that's protected from noise."

"And the problem?" asked Polly.

"When I have three qubits, the combination is straightforward — like on the Island of Threes. But when I try to verify that the encoding worked correctly, I have to check four things simultaneously." He held up four fingers. "The four stabiliser measurements of my error-correcting code. And checking them in the wrong order gives wrong answers."

"Because you're checking four things," said Polly, "and four things form a tetrahedron."

"Exactly. The four checks sit at the vertices. The 6j symbol — which in my case is just plus-one or minus-one, because I'm working with simple binary spins — tells me whether my checks are consistent with each other. If the 6j symbol is minus one on a particular edge, I have to flip a sign when I switch between two checking schemes. Get the sign wrong and I think I have an error when I don't, or think I'm fine when I'm not."

He pressed a button. The box chirped.

"It's the same tetrahedron as nuclear physics, as quantum gravity, as conformal field theory," said Mr Tomkins, slowly. "You're all asking the same question."

"What is the amplitude for changing the order of four things," said Polly. "Everyone who ever asks that question gets the same answer: a tetrahedron, a 6j symbol, and the pentagon equation lurking in the background."

Chapter Seven: The Sleeping Astronomer

Near the top of the island's only hill was an observatory. Inside, an astronomer was asleep in her chair with a star map draped over her face.

Polly lifted a corner of the star map gently, and underneath was a diagram of three small dots orbiting each other in a complex figure-eight pattern.

"Three bodies?" asked Mr Tomkins.

"Three bodies," said Polly. "The three-body gravitational problem. Three stars orbiting each other in a stable choreography. And here's the thing:" she pointed to a number written in the margin, "the stable orbits exist only when a certain counting number — 168 — divides evenly into the orbit's period."

"One hundred and sixty-eight. Why that?"

"Because 168 is the size of a particular symmetry group — PSL(2,7), the automorphism group of the Fano plane. Which is the same symmetry group that controls the 6j symbols for three-qubit quantum systems. Which is why the same tetrahedra that appear in quantum information also determine which three-body orbits are stable."

Mr Tomkins looked at the sleeping astronomer. "She doesn't know that her three-star system is secretly controlled by a quantum information calculation?"

"She knows about the stability condition. She doesn't know it's a 6j symbol. But it is. The universe is consistent: every time four things interact — whether they're nucleons, quarks, quantum bits, spacetime chunks, or stars in a gravitational ballet — the same tetrahedron appears and asks the same question."

Chapter Eight: The Pentagon Rule, Explained Simply

Outside the observatory was a wide flat terrace, and on the terrace a teacher had drawn five large tetrahedra in chalk, arranged in a ring. A class of students was examining them.

"Imagine," said the teacher, "that you're building a crystal — a solid with a regular repeating structure. Each unit of the crystal is a tetrahedron.

Each face of one tetrahedron is glued to a face of the next."

She walked around the ring.

"Now imagine you're an ant walking from one tetrahedron to the next, carrying a number — the amplitude for whichever recoupling happened in that tetrahedron. As you walk, you multiply your number by the next 6j symbol at each step."

The students nodded.

"If you walk around the whole ring and come back to where you started, what number do you have?"

A student said: "It depends on how many tetrahedra you crossed."

"It depends on the 6j symbols you multiplied together," said the teacher. "Now here is the key. In a consistent theory — one where the physics doesn't depend on which path you take — you must get the same answer regardless of which sequence of tetrahedra you cross. If you go clockwise or anti-clockwise, short-cut or the long way around, you must get the same number."

"And the pentagon equation?" asked Polly.

"Is the condition that guarantees this. The simplest non-trivial loop contains exactly five tetrahedra — five 6j symbols in a ring. The pentagon equation says: the product of these five 6j symbols, in the right order, equals the identity. One. Round-trip."

She wrote it on the board:

$$F_{12} \cdot F_{23} \cdot F_{34} \cdot F_{45} \cdot F_{51} = 1$$

"Each F is a 6j symbol — a change of coupling order for four objects. Five of them in a ring. The pentagon equation says their product is trivial. If it is, the theory is self-consistent: you can trust the physics regardless of the order you combine things. If it isn't, the theory has a fundamental contradiction."

"And what kinds of theories satisfy it?" asked Mr Tomkins.

"All the well-behaved ones," said the teacher. "Quantum mechanics, general relativity, the Standard Model of particle physics. They all satisfy the pentagon equation. That's part of what makes them work. Theories that don't satisfy it have internal contradictions — you'd get different answers for the same physical question depending on which way you computed it."

She paused. "And this is also why, in a quantum computer, you can verify whether a prover is honest. A dishonest prover — one who doesn't actually hold the quantum state they claim to hold — cannot satisfy all five conditions of the pentagon equation simultaneously. The pentagon is a consistency test that a cheat cannot pass."

Chapter Nine: What Three Can't Do and Four Can

On the way back down to the beach, Mr Tomkins put it all together in his head.

"So let me see if I understand," he said. "With two things, you can only combine them one way: A plus B."

"Correct," said Polly.

"With three things, you can combine them two ways — A-with-B first, or B-with-C first — but the two ways always give the same answer. No new information."

"Correct."

"With four things, for the first time, the two different ways of combining give different answers. The difference is measured by the 6j symbol — six numbers arranged on the edges of a tetrahedron. And the 6j symbol is the same number whether you're doing nuclear physics, quantum gravity, knot theory, quantum error correction, financial risk, or three-body astronomy."

"Correct."

"And the pentagon equation is the condition that says: even though individual 6j symbols are non-trivial, any closed loop of them adds up to nothing. Consistency survives."

"Exactly right," said Polly. She smiled. "You've understood the whole paper in a dream."

"And the things that *can't* be fixed by local checking — the problems that only appear when all four objects are considered together — those are the H^2 obstructions?"

"Now you're ahead of the paper," said Polly. "Yes. The tetrahedron is the minimal shape that can carry a second cohomology class — a global inconsistency that is invisible from any triangle, any edge, any vertex. You can check every pair, every trio, and miss the problem entirely. Only when you check all four simultaneously does it show up. That's the deep reason why four is the magic number."

Mr Tomkins stopped walking.

Behind them, the Island of Threes was peaceful and orderly, its tables all carrying their tidy triplets. Ahead, the Island of Fours shimmered with a slightly more complex light — the light of a universe where combining things in different orders gives different results, and where you need a tetrahedron to keep track of the difference.

"Can you go beyond four?" he asked.

"Of course," said Polly. "Five objects gives you the pentagon equation. Six objects gives you what they call the 9j symbol. But here's the beautiful thing: none of those are genuinely new. Every 9j symbol decomposes into a sum of products of 6j symbols. Every higher combination decomposes too. Four is the last primitive — the last new type of mathematical object you ever need. After that it's all software. Programs written in the language of tetrahedra."

Mr Tomkins looked at his hand. He spread four fingers.

"Four is where things start being genuinely complicated," he said. "And the tetrahedron is the shape that carries the complication."

"The universe is full of tetrahedra," said Polly. "Once you know where to look."

Epilogue: The Lecture, Again

Mr Tomkins woke up with a start.

The professor at the front was still writing on the board. She had moved on from the curly brackets to a drawing — a triangular pyramid with six labels on its edges.

A tetrahedron.

Mr Tomkins sat up very straight. For the first time in a long career of falling asleep in lectures, he understood exactly what was on the board. He understood why six labels, why a tetrahedron, why four objects and not three, why the pentagon equation appeared as a consistency condition, and why minus one-third kept showing up in nuclear physics and particle physics and quantum information, all from the same mathematical source.

He raised his hand.

The professor looked surprised — nobody ever asked questions from the back row.

"Yes?" she said.

"The tetrahedron," said Mr Tomkins, "is the shape of the simplest thing you can't figure out by checking parts separately. You need all four at once. Is that right?"

The professor stared at him for a long moment.

"That," she said, "is the most concise summary of the 6j symbol I have ever heard."

Mr Tomkins smiled and settled back in his chair.

He did not fall asleep again.

The End

A Note for Teachers

The ideas in this story correspond directly to mathematical structures in the research paper *In Praise of Tetrahedra* (Buckley, 2026). The mapping is as follows:

Story element	Mathematical concept
Island of Threes / no ambiguity	Associativity of tensor products; Clebsch-Gordan coefficients
Island of Fours / three schemes	Three bracketing schemes for four representations
6j symbol as a tetrahedron	Wigner 6j symbol; Ponzano-Regge observation
Raj's shells giving different results	Non-trivial recoupling at n=4
Mr Vier's bank failure	H^2 cohomological obstruction; financial pentagon identity
Dr Pandya's nuclear recoupling	Pandya transformation (same −1/3 in nuclear and charmonium)
Kieran's quantum error correction	Stabiliser codes; Fano 6j values ±1
The sleeping astronomer (168)	Three-body orbit stability;
Pentagon game on the terrace	Pentagon equation; MacLane coherence; MIP* verifier constraint
"Four is the last primitive"	6j completeness theorem (all higher nj = sums of products of 6j)

The key pedagogical points:

1. **Three is never enough, four is always enough.** Two and three objects have trivial recoupling. Four is the minimum at which order matters and a non-trivial amplitude (the 6j symbol) is required.
2. **The tetrahedron is not a metaphor.** The 6j symbol genuinely has the symmetry group of the tetrahedron (A_4), and the six entries are literally the six edges of a tetrahedron. The same formula appears identically in all the domains listed.
3. **The pentagon equation is the consistency condition.** It says that the result of any sequence of reorderings depends only on start and end, not on the path. This is simultaneously the Biedenhahn-Elliott identity of spectroscopy, the coherence condition of fusion categories, and the MIP* verifier protocol.
4. **Local checks miss global problems.** The financial example (Mr Vier) illustrates that pairwise and triangular consistency is insufficient.

The tetrahedron is the minimum shape that can carry an H^2 obstruction — a global inconsistency invisible to all local checks.